

Arithmetic of the integer quantum Hall effect

Vipin Srivastava

School of Physics, University of Hyderabad, Hyderabad - 500 046, India

Integer quantum Hall effect (IQHE) has been analysed considering the degeneracies of localized and extended states separately. Occupied localized and extended states are counted and their variation is studied as a function of magnetic field. The number of current carrying electrons is found to have a saw-tooth variation with magnetic field. The analysis attempts to answer certain basic questions besides providing a simple but complete understanding of IQHE.

PACS Nos.: 73.40.Hm, 72.15.Rn

We show that in the integer-quantum-Hall setting the number of current carrying electrons varies like saw-tooth with the magnetic field. Infact we find that this is an alternative manifestation of the integer-Hall-quantization.^{1,2} We also suggest an experiment for counting the number of extended and localized states below the Fermi level as a function of magnetic field B . Besides revealing some more interesting physics embedded in the phenomenon of integer-quantum-Hall-effect (IQHE) and providing the simplest way of understanding the fascinating phenomenon, the present approach to the IQHE is expected to resolve, through the suggested experiment, the following long standing questions: (A) How does the IQHE approach the 2-dimensional localization result — localization of all states at any disorder³ — in the limit $B \rightarrow 0$? One has to resolve between two apparently possible alternative scenarios, namely (i) the extended states ‘float up’ to infinite energy as $B \rightarrow 0$;⁴ and (ii) the critical disorder W_c , required to localize all states in a band approaches zero as $B \rightarrow 0$.⁵ (B) Whether the number of extended states in a Landau subband forms a vanishing or a non-vanishing fraction of the total number of states in the subband? We have also addressed two questions related to the basic understanding to the IQHE: (C) How does the IQHE acquire the spectacular accuracy and what are the factors that put limit on it? And (D) How is it that exactly $ls(B)$ states ($s(B)$ being the degeneracy of a Landau subband) play the central role in the integer Hall quantization⁶ although all $ls(B)$ states may not be occupied, or the number of occupied states may far exceed $ls(B)$ for a value of B at which the Fermi level E_F is located in the l^{th} mobility gap?

We will count the number of extended and localized electrons as a function of B , first assuming the Landau subbands to be independent, and then by incorporating the result of Haldane and Yang⁴ to discuss the effect of band-mixing.

Take $B = 0$ to start with and consider an increase in B by δB that inserts *one* flux

quantum into the system. There will be N Landau levels below E_F in a system of N electrons per unit area, and each level will have one state (spin is not important for our purpose). Due to the presence of disorder we ask: are the N states (a) all localized; or (b) all extended or (c) some localized and others extended? Neither (a) nor (b) can hold *as a rule*, for then *all* subsequent increments of B by δB would introduce either only localized (in case (a)) or only extended states (in case (b)) and consequently all states in the system would be either localized or extended at all $B > 0$. Both these possibilities are contrary to the known results. Therefore, (c) must represent the true situation. Now the question arises: as an increment δB adds a new state to each Landau subband the fractions of the *new* states below E_F that are respectively localized and extended decided arbitrarily or is there a rule governing it? We expect an underlying rule connected with the fact that the amount of localization is decided by the strength of disorder. So, for each Landau subband we should be able to write,

$$(\text{no. of localized states})/(\text{no. of extended states}) = D, \quad (1)$$

which besides depending on the strength of disorder should depend on B as well.

The arithmetic: Recall that classically (without disorder) the Hall voltage can be written as

$$\mathcal{E}_y(B) = s(B)vh/e, \quad (2)$$

where $s(B)$ is the degeneracy of each Landau level and v is the average drift velocity of current carriers. In the presence of disorder and localization we split $s(B)$ as

$$s(B) = s^E(B) + s^L(B), \quad (3)$$

with E and L respectively representing extended and localized states, and write the Hall voltage in analogy with (2) as

$$E_y(B) = s^E(B)V(B)h/e, \quad (4)$$

keeping the system current density $j_x = n^E(B)eV(B)$ carried by n^E extended electrons unchanged at the value Nev (as in a typical IQHE experiment). The constancy of j_x leads to

$$V(B) = (N/n^E(B))v , \quad (5)$$

that is, n^E electrons per unit area carry the current Nev by moving at a higher drift velocity V to compensate for the loss of current due to localization of $n^L (= N - n^E)$ electrons.

The $s^E(B)$ in a particular band always increases with B though non-monotonically — it goes up by 1 only when δB -increase of B adds an extended state to this band which happens with probability $1/(D+1)$ in view of (1) (note that following (3), eqn.(1) will become $s^L(B)/s^E(B) = D$). But we will see that $V(B)$ increases as well as decreases with B depending on where E_F is located. So, $E_y(B)$ can remain unchanged with B whenever $V(B)$ decreases, in case

$$s^E(B)V(B) = \text{a constant} . \quad (6)$$

We will count the occupied localized and extended states as a function of B and investigate the quantum Hall plateaus through (6) and address the questions stated above. We will follow the picture of Fig.1 commonly used in connection with IQHE.⁷

Suppose E_F is located in the l^{th} mobility gap and the numbers of occupied extended and localized states are respectively $ls^E(B)$ and $ls^L(B) + \eta$ (see Fig.1d for η), so that $V(B) = [\{ls(B) + \eta\}/\{ls^E(B)\}]v$. Now B is increased by δB , and l new states — one each in l subbands below E_F — are added. Suppose i of these states are extended and $(l - i)$ localized. The E_F will move downwards by l states and the numbers of extended and localized states will become $(ls^E(B) + i)$ and $(ls^L(B) + \eta - i)$ respectively. Then,

$$s^E(B + \delta B)V(B + \delta B) = s^E(B + \delta B) \frac{ls(B) + \eta}{ls^E(B) + i} v = \frac{s^E(B + \delta B)}{s^E(B) + i/l} s^E(B)V(B) . \quad (7)$$

And, if E_F lies in the l^{th} band of extended states, then the counting of localized and extended states below E_F would give

$$s^E(B + \delta B)V(B + \delta B) = \frac{s^E(B + \delta B)}{s^E(B) - (l - i)/l} s^E(B)V(B) \quad . \quad (8)$$

To get the behaviour of E_y we will examine

$$\frac{\delta E_y(B)}{E_y} = \frac{[s^E(B + \delta B) - s^E(B)] - i/l}{s^E(B) + i/l} \text{ for } E_F \text{ in } l^{\text{th}} \text{ mobility gap;} \quad (9a)$$

and,

$$= \frac{1 + [s^E(B + \delta B) - s^E(B)] - i/l}{s^E(B) - (l - i)/l} \text{ for } E_F \text{ in } l^{\text{th}} \text{ band of ext.sts.;} \quad (9b)$$

Note that i can be 1 only with probability $l/(D + 1)$, and that $[s^E(B + \delta B) - s^E(B)]$ can only be either 0 or 1 for a subband since s^E should be an integer. So, in (9a) δE_y remains zero until the magnetic field is incremented by $[(D + 1)/l]\delta B$ when i becomes 1 with probability one and the s^E becomes $s^E(B) + 1$ in one of the l subbands, and stays at value $s^E(B)$ in the remaining $(l - 1)$ subbands. The subband that gets the new extended state makes a non-zero contribution to (9a). This makes δE_y non-zero of order $[10^5 s(B)/(D + 1)]^{-1}$.⁸ The inaccuracy δE_y remains the same on the further increase of magnetic field until the next extended state is introduced below E_F . In this way a plateau is formed in the E_y with an accuracy of few parts in $[10^6 s(E)/(D + 1)]$.

For the $V(B)$ note that when E_F lies in a mobility gap we will have either $V(B + \delta B) < V(B)$ whenever an extended state is produced and the s^E is enhanced in the subbands below E_F , or $V(B + \delta B) = V(B)$ inbetween these events. On the other hand when E_F lies in a band of extended states we will always have $V(B + \delta B) > V(B)$ because $n^E(B)$ will necessarily decrease due to the downward movement of E_F . We find here that good amount of information can be extracted from the variation of $V(B)$ with B . Before we

go into the details of the variation of $V(B)$ we will understand the role played by the flexibility of $V(B)$ in the light of the question (D).

If the given N electrons *exactly* fill l levels then from (2), in the *classical case*

$$R_H = \frac{\mathcal{E}_y(B)}{j_x} = \frac{h}{le^2} , \quad (10)$$

and this result can be maintained as independent of N and B *classically* by adding l electrons to the system from outside each time B is increased by δB , and by maintaining j_x at Nev (which reduces v suitably as $N \rightarrow N + l$). The IQHE presents a setting where the system, under certain conditions, on its own mimics this classical scenario — quantum localization of electrons creates a buffer of states which feeds electrons to l completely filled bands of extended electrons, and keeps them completely filled over a range of B . As long as the bands of current - carrying electrons are exactly filled and j_x is maintained constant, the number of electrons in the bands has no relevance, only the number of bands matters for R_H as in the above classical case. The exact filling of l bands of extended electrons is therefore exactly equivalent to the exact filling of l Landau subbands (with both localized and extended states in them) as well as l Landau levels in the classical case (i.e., without localization). In such a situation with the help of (4) we have

$$j_x = Nev = ls^E(B)eV(B) = E_y(B)le^2/h \equiv ls(B)e\frac{E_y(B)}{B} , \quad (11a)$$

so that

$$R_H = \frac{E_y(B)}{j_x} = \frac{B}{ls(B)e} = \frac{h}{le^2} ; \quad B \in (B_a, B_b), \quad \text{say} . \quad (11b)$$

Thus *all* the states in l subbands, $ls(B)$, and *only these many* states matter when the Hall effect is quantized irrespective of the facts that N may be $<$ or even $>$ $ls(B)$.

Returning to $V(B)$ we note that it oscillates about $(D + 1)v$. When E_F is in the l^{th}

mobility gap,

$$V(B) = \frac{ls(B) \pm \eta}{ls^E(B)} v = (D+1)v \pm \frac{\eta}{ls^E(B)} v ; \quad \eta \geq 0, \quad (12a)$$

i.e., it decreases from above $(D+1)v$ to below it as B increases. And for E_F lying in the l^{th} band of extended states,

$$V(B) = (D+1)v + \frac{\frac{1}{2} - f}{l-1+f} Dv ; \quad 0 \leq f \leq 1, \quad (12b)$$

where f is the occupation fraction of the upper most band of extended states; so $V(B)$ increases from below $(D+1)v$ (for $f \sim 1$) to above it (for $f \sim 0$) as B increases. $V(B) = (D+1)v$ for $\eta = 0$ and $f = \frac{1}{2}$.

Since $j_x = Nev = n^E(B)eV(B)$, we have

$$n^E(B) = \frac{N}{D+1 \pm \eta/(ls^E(B))} \quad \text{for } E_F \text{ in } l^{\text{th}} \text{ mobility gap ;} \quad (13a)$$

and

$$= \frac{N}{D+1 + \frac{\frac{1}{2}-f}{l-1+f}} \quad \text{for } E_F \text{ in } l^{\text{th}} \text{ band of ext.sts. .} \quad (13b)$$

The $n^E(B)$ oscillates about $N/(D+1)$, the value it attains when $\eta = 0$ and $f = 1/2$.

To plot $V(B)$ and $n^E(B)$ we make following additional observations with reference to Fig. 1(c):

(i) $V(B_a) - V(B_l) = V(B_l) - V(B_b)$, since

$$V(B_a) = [1 + D/\{2l(D+1)\}]V(B_l) , \text{ and} \quad (14a)$$

$$V(B_b) = [1 - D/\{2l(D+1)\}]V(B_l) . \quad (14b)$$

(ii) $V(B_a) < V(B_c)$ since

$$\frac{V(B_a)}{V(B_c)} = \frac{l-1}{l} \frac{2l(D+1) + D}{2(l-1)(D+1) + D} < 1 . \quad (15)$$

(iii) The number of localized states scanned when E_F moves from its position at B_a to that at B_l is $s^L(B_a)/2$, and it is $s^L(B_l)/2$ when E_F goes from B_l to B_b . Since $s^L(B_l) > s^L(B_a)$ the plateau must be *asymmetric* about the classical $R_H(B)$ – line even under the ideal conditions of symmetric subbands.

The saw-tooth variation of $V(B)$ is shown schematically in Fig. 2(a). The $n^E(B)$ varies in a manner complementary to that of $V(B)$ — Fig. 2(b). The bend in each arm of variation is due to the combined effects of (iii) and (i). The $V(B)$ and $n^E(B)$ will approach finite non-zero values, $(D+1)v$ and $N/(D+1)$ respectively, in the $B \rightarrow 0$ limit if D is assumed to be independent of B .

However, D must diverge as $B \rightarrow 0$ if $n^E(B)$ must approach zero in this limit to yield the well known $2d$ localization result.³ That is, the $(D+1)v$ –, and $N/(D+1)$ – lines about which $V(B)$ and $n^E(B)$ oscillate should indeed stoop upwards and downwards respectively as shown. In the case of $n^E(B)$ the $N/(D+1)$ – line can meet the B – axis either at $B = 0$ or at a $B > 0$. The former would correspond to the possibility discussed in the set of references (5) — $n^E(B)$, on average, would decrease with B , becoming zero only at $B = 0$; so the amount of extra disorder required to convert them into localized states too would approach zero as $B \rightarrow 0$, i.e., $W_c(B) \rightarrow 0$ as $B \rightarrow 0$.⁵ On the other hand if the band-mixing, studied by Haldane and Yang,⁴ has to have a noticeable effect to lead to the floatation of extended states as proposed by Khmel'nitskii, and Laughlin,⁴ then the $N/(D+1)$ – line should be expected to converge with decreasing B to a point at $B > 0$ — since the band-mixing causes the energies of extended states to shift upwards, as shown by Haldane and Yang,⁴ with decreasing B besides decreasing in number, the extended states should also be moving steadily from below the E_F to above it, so their number below the E_F should deplete faster than in the previous case and the region below the E_F should become devoid of extended states well before $B = 0$ is reached.

The rate at which the D diverges as $B \rightarrow 0$, which we need to know in order to resolve between the two situations discussed above, can be determined from the following experiment.

Suggested experiment: Within the usual IQHE set up we propose the following to count the number of occupied localized and extended states at a given value of B in a sample of *known* N . Set B at the value, say B_a , corresponding to the beginning of a plateau, say $l = 2$, and reduce the number-density of electrons from the initial value N by changing the gate voltage while keeping the j_x fixed at $Ne v$ and B at B_a . This will move the E_F towards the point $B = B_2 (\equiv B_{l=2})$ of Fig.1c. The quantum Hall voltage $E_y(B_a)$ will not change in this process but the classical Hall voltage $\mathcal{E}_y(B_a) (= B_a/(Ne))$ will increase. By monitoring the variation of \mathcal{E}_y the E_F can be moved to the position corresponding to $B = B_2$ where $\mathcal{E}_y(B_a)$ will become equal to $E_y(B_a) (= h/(2e^2))$. Determine the number-density of electrons at this stage. Suppose it is N' . Then N' will be the number of electrons filling two subbands *exactly* and the electrons removed from the system, $N - N'$, will be from the localized states. So, $2(N - N')$ will be the number of *localized* electrons per subband at $B = B_a$, and we will have

$$S^E(B_a) = [N' - 4(N - N')]/2 \quad , \text{ and} \quad (16)$$

$$D(B_a) = 4(N - N')/(5N' - 4N) \quad . \quad (17)$$

The $D(B_2)$ can be similarly determined. The $D(B_b)$ too can be determined in the above way but by adding the electrons to the empty localized states in the upper half of the subband $l = 2$. In this way even without knowing the density of states we can measure $D(B)$ at certain special values of B (such as B_a, B_2, B_b, B^2 etc.) and produce the salient features of the $n^E(B)$ - graph. An experimental set up good enough to produce sufficiently precise large- l plateaus should enable us to see whether the $N/(D + 1)$ -line meets the

B -axis at $B = 0$ or at a $B > 0$.

Finally, the answer to question (B) is apparent from the present analysis — the number of extended states in a Landau subband form a vanishing fraction of the total number of states in it only in the limit $B \rightarrow 0$ when $D \rightarrow \infty$, otherwise this ratio is always non-vanishing. This however does not contradict the possibility of all the extended states occurring at a single energy in the centre of a subband.⁹

In summary, simply by splitting $s(B)$ into $s^E(B)$ and $s^L(B)$ and writing $E_y(B)$ in terms of $s^E(B)$ and $V(B)$ we are able to translate the IQHE result in terms of $V(B)$ and $n^E(B)$ which are found to have novel saw-tooth variations as a function of B . The proposed simple extension of the IQHE experiment to measure $n^E(B)$ can resolve the controversy about the approach of the IQHE to the $2d$ localization result in the limit $B \rightarrow 0$. The present alternative view of the IQHE result also provides an easy understanding of the phenomenon.

Acknowledgements: I am grateful to Prof.Sir Sam Edwards for discussions and hospitality at the Cavendish Laboratory where this work was initiated. Thanks are also due to Prof.D.Shoenberg for discussions. Financial support for this work was provided by the Leverhulme Trust, London and the Association of Commonwealth Universities.

References

1. K. von Klitzing, G. Dorda and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
2. D. Shoenberg, *Magnetic Oscillations in Metals* (Cambridge University Press, 1984) also pointed out (p. 159) that the IQHE result¹ should imply a saw-tooth variation of the number of mobile electrons with magnetic field but did not give a mechanism for it.
3. E. Abraham, P.W. Anderson, D.C. Licciardello and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
4. D.E. Khmelnitskii, Phys. Lett. **106A**, 182 (1984); R.B. Laughlin, Phys. Rev. Lett. **52**, 2304 (1984); K. Yang and R.N. Bhatt, *ibid* **76**, 1316 (1996); F.D.M. Haldane and K. Yang, *ibid* **78**, 298 (1997) and references therein.
5. D.Z. Liu, X.C. Xie and Q. Niu, Phys. Rev. Lett. **76**, 975 (1996); D.N. Sheng and Z.Y. Weng, *ibid* **78**, 318 (1997), and references therein.
6. The Hall voltage $E_y(B) = h/(le^2)j_x = B/[ls(B)e]j_x$ along the l^{th} plateau with the system current j_x fixed at Nev (N = no. density of electrons in the system; v = drift velocity). Note that exactly $ls(B)$ states matter eventhough N may be $>$ or even $< ls(B)$.
7. R.B. Laughlin, Phys. Rev. B**23**, 5632 (1981).
8. The E_y is typically of order 10^{-5} ; $s^E(B) = s(B)/(D + 1)$ where $s(B) \sim 10^9$ per mm for $B \sim 10$ T.
9. See e.g., Y. Huo and R.N. Bhatt, Phys. Rev. Lett. **68**, 1375 (1992).

Figure Captions:

Figure.1:

(a) The integer quantum Hall effect geometry; (b) the IQHE plateaus in the Hall voltage (for a fixed system current j_x); broken line shows the classical Hall effect result; (c) density of states (DOS) comprising disorder-broadened Landau levels with extended states in the middle and localized states in the shaded regions; (d) enlargement of a portion of (c) — the cross-hatched region has η localized states.

Figure.2:

Schematic representation of the saw-tooth variation of (a) drift velocity $V(B)$, and (b) number of extended electrons $n^E(B)$. The oscillations happen in (a) and (b) respectively about $(D+1)v-$ and $N/(D+1)-$ lines where \cdots and $-$ represent $D \rightarrow \infty$ without and with band mixing.